

Gravitational Tension and Thermodynamics of Planar AdS Spacetimes

Basem Mahmoud El-Menoufi¹, Benjamin Ett², David Kastor³, Jennie Traschen⁴

Department of Physics
University of Massachusetts
Amherst, MA 01003

Abstract

We derive new thermodynamic relations for asymptotically planar AdS black hole and soliton solutions. In addition to the ADM mass, these spacetimes are characterized by gravitational tensions in each of the planar spatial directions. We show that with planar AdS asymptotics, the sum of the ADM mass and tensions necessarily vanishes, as one would expect from the AdS /CFT correspondence. Each Killing vector of such a spacetime leads to a Smarr formula relating the ADM mass and tensions, the black hole horizon and soliton bubble areas, and a set of thermodynamic volumes that arise due to the non-vanishing cosmological constant. These Smarr relations display an interesting symmetry between black holes and bubbles, being invariant under the simultaneous interchange of the mass and black hole horizon area with the tension and soliton bubble area. This property may indicate a symmetry between the confining and deconfined phases of the dual gauge theory.

¹bmahmoud@physics.umass.edu

²bett@physics.umass.edu

³kastor@physics.umass.edu

⁴traschen@physics.umass.edu

1 Introduction

This paper will explore certain aspects of the thermodynamics of asymptotically planar AdS spacetimes, such as the planar black hole [1, 2, 3] and soliton [3] solutions studied in these references. Through the gauge/gravity correspondence, the thermodynamics of such spacetimes are known to have important physical consequences. For example, it was argued in [3] following [4, 5] that the planar AdS black hole is dual to the high temperature, deconfined phase of the associated strongly coupled gauge theory on the AdS boundary, with the AdS soliton corresponding to the low temperature, confining phase of the theory. Further support for this interpretation comes from the values of the Euclidean actions, or free energies, of the black hole [6, 7] and the soliton [7] which demonstrate that the system indeed undergoes a phase transition as in [8]. Further studies of *e.g.* the approach to equilibrium in this system have appeared in [9, 10, 11, 12].

The gravitational charges of planar AdS spacetimes display interesting features, with the AdS soliton having a negative, but nonetheless physical, mass [3]. Our exploration begins with the observation that the mass does not fully characterize the asymptotic behavior of such spacetimes. Due to the planar spatial directions at infinity one expects that, as in the case for Kaluza-Klein black holes [13, 14, 15, 16], gravitational tensions should also make important contributions to basic thermodynamic relations. Indeed, the tracelessness of the boundary CFT stress tensor implies that tension or pressure⁵ must play an important role. Consistent with this, we will see that the AdS soliton, with its negative mass, has a large positive tension, with this situation being reversed for the black hole.

With nonzero Λ another important ingredient enters the thermodynamic relations as well. It has been shown that to derive a Smarr formula for (A)dS black holes, one must include a new potential V , which is thermodynamically conjugate to the cosmological constant in the sense that it gives the change in mass when Λ is varied [17]. This new potential V has the dimensions of volume, and following [18] we will refer to it as the thermodynamic volume. Regarding the cosmological constant as a pressure, the Smarr formula for (A)dS black holes then contains a pressure-volume type contribution proportional to ΛV . Computations of V and studies of associated thermodynamic phenomena have been carried out in [17, 18, 19, 20, 21, 22] for AdS black holes with spherical topology, and in [23] for the dS case.

Smarr formulas for gravitational systems are analogous to the Gibbs-Duhem relations of classical thermodynamics. In the gravitational case, a Smarr formula is an exact relation that holds for stationary black holes, and more generally for any spacetime having a Killing symmetry, that relates the ADM gravitational charges [24] defined in an asymptotic region to geometric properties of the interior of the spacetime, such as the area and surface gravity of a black hole horizon. Originally derived for four dimensional, stationary, asymptotically flat black holes (*i.e.* black holes of astrophysical interest) in [25], extensions of these relations have been made to higher spatial dimensions, different horizon topologies, higher derivative theories of gravity, and to spacetimes with different asymptotics.

We will derive Smarr relations for asymptotically (locally) AdS spacetimes with planar

⁵Tension is defined as the negative of pressure.

spatial cross sections, *i.e.* such that the boundary at infinity is a plane or torus cross time. The ADM charges for these spacetimes are the mass and spatial tensions together with the linear and angular momenta. We will focus on static spacetimes, with vanishing linear and angular momenta, containing black holes and/or bubbles. This problem appears at first sight to be a simple generalization of two previously considered cases. Black holes and bubbles with compact spatial directions (but $\Lambda = 0$) were considered in [14, 26, 16, 13, 15], while spherical AdS black holes (*i.e.* with $\Lambda \neq 0$ but no compact spatial directions) were considered in [27, 28]. The compact spatial directions introduce new terms in the Smarr relations of the form $L\mathcal{T}$, with L being the length of a given compact spatial direction and \mathcal{T} being the gravitational tension in that direction, while as noted above taking $\Lambda \neq 0$ introduces a ΛV term. We will see, however, that combining compact spatial dimensions with $\Lambda \neq 0$ is in fact not just a simple addition problem, but rather involves interesting new features.

The most prominent new feature that arises is a tracelessness condition

$$\mathcal{M} + \sum_i L_i \mathcal{T}_i = 0 \quad (1)$$

satisfied by the ADM mass and tensions, where the index i runs over the planar spatial directions. While a condition of this sort would naturally be expected from the AdS/CFT standpoint, it is interesting to see how it arises on the gravity side in terms of standard ADM conserved charges [24]. The condition (1) also helps make sense of unusual signs that arise for the mass and tensions with $\Lambda < 0$. In the asymptotically flat case, the ADM mass is known to be positive [29, 30], and the gravitational tensions have also been shown to be positive [31] for static spacetimes. However, as we will see, in AdS there are examples with positive mass but negative tensions, and similarly solutions with positive tensions but negative mass. Equation (1) makes it clear this is to be expected in AdS. All the gravitational charges cannot be positive. The constraint (1) also implies that the ADM charges are not all independent, which will have practical implications in our construction. In particular, the inversion of Komar integral identities necessary to obtain the Smarr results will involve technical complications, dealt with in section (3.1) below, that were not present in the $\Lambda = 0$ case.

Our strategy in the paper is as follows. We study solutions to the Einstein equation with negative cosmological constant

$$R_{ab} = \frac{2\Lambda}{D-2} g_{ab} \quad (2)$$

where $\Lambda = -(D-1)(D-2)/2l^2$. We will assume that the spacetime is static and has p spatial translation Killing vectors with $0 \leq p \leq D-2$. The planar black hole and AdS soliton solutions studied in [3] both have $D-2$ spatial translation symmetries. We will also assume that there are no shear terms at infinity, so that asymptotically the metric has the form

$$ds^2 \simeq \frac{r^2}{l^2} \left(\eta_{\alpha\beta} dx^\alpha dx^\beta + \sum_{\alpha=0}^{D-2} \frac{c_\alpha}{r^{D-1}} (dx^\alpha)^2 \right) + \frac{l^2}{r^2} \left(1 + \frac{c_r}{r^{D-1}} \right) dr^2 \quad (3)$$

where the c_α 's and c_r are constant fall-off coefficients, and we take the planar spatial directions x^i with $i = 1, \dots, D-2$ to be periodically identified according to $x^i \equiv x^i + L_i$. We assume that $p+1$ of the vector fields $\xi_{(\beta)} = \partial/\partial x^\beta$, including necessarily $\xi_{(0)}$, satisfy Killing's equations throughout the spacetime. Each of these Killing vectors leads to a Komar integral identity, which provide relations between the the fall-off coefficients c_α , and geometric properties of black hole horizons and/or soliton bubbles in the interior, plus terms of the form $\Lambda V_{(\beta)}$, where $V_{(\beta)}$ is the thermodynamic volume associated with the Killing vector $\xi_{(\beta)}$. After taking into account the trace constraint (1) and a related gauge fixing, the fall-off coefficients c_α may be solved for in terms of the ADM charges giving the Smarr relations.

There is a geometric distinction between spacelike Killing vectors that act freely and those that have one or more fixed points, referred to as 'bubbles' (see *e.g.* [14, 26, 13] for related discussions of bubbles with $\Lambda = 0$). Gravitational tensions then divide into three categories; tensions associated with freely acting symmetry directions, which will be denoted by \mathcal{T}_F ; tensions associated with non-freely acting symmetry directions, which will be denoted⁶ by \mathcal{T}_B ; and those which are not along a symmetry direction. The asymptotic lengths of these dimensions will similarly be denoted by L_F and L_B . The Smarr relations are simplest when all the Killing vectors commute, in which case we will show that all the thermodynamic volumes are equal, $V_{(\beta)} = V$, and the Smarr relations reduce to

$$\mathcal{M} = \frac{\kappa_H A_H}{8\pi} + \frac{\Lambda V}{8\pi(D-2)} \quad (4)$$

$$L_B \mathcal{T}_B = \frac{\kappa_B L_B A_B}{8\pi} + \frac{\Lambda V}{8\pi(D-2)} \quad (5)$$

$$L_F \mathcal{T}_F = \frac{\Lambda V}{8\pi(D-2)} \quad (6)$$

where the surface gravity for a smooth bubble is given by $\kappa_B L_B = 2\pi$ (see [16] for further discussion). The formula for \mathcal{T}_F is a special case of \mathcal{T}_B with the bubble area equal to zero. In this case the formula says that the globally defined gravitational tension times the length, $L_F \mathcal{T}_F$, is proportional to minus the intrinsic pressure times the thermodynamic volume V , where the intrinsic pressure is due to the cosmological constant with $P \propto -\Lambda$. The Smarr formulas display an interesting symmetry between black holes and bubbles, since they are invariant under interchanging the contributions from the mass and horizon area with the contributions of the bubble tension and bubble area. These equations also raise the intriguing possibility of a critical solution that is a mixture of bubbles and black holes, representing a transition between the high and low temperature phases of the field theory, which would have $\mathcal{M} = \mathcal{T}_B = 0$, $\kappa_H A_H/8\pi = \kappa_B A_B/8\pi = -\Lambda V/8\pi(D-2)$, and positive tensions which are not associated with a symmetry to satisfy the trace constraint (3).

This paper is organized as follows. In Section (2) we use the ADM formalism to compute the mass and gravitational tensions in terms of the assumed asymptotic form of the metric (3). In Section (3) we show how to take into account the trace condition (1) amongst the

⁶Further notation will be added as needed below to distinguish between different freely, or non-freely, acting directions.

gravitational charges and deal with a related gauge fixing issue. We also compare the ADM charges to the results of boundary stress tensor computations. The Smarr relations are derived in Section (4) and the results are collected in Section (5) along with comments on open questions. The Appendix contains a derivation. We use the notational conventions that Greek indices $\alpha, \beta = 0, \dots, D - 2$ label directions tangent to the planes of constant radial coordinate r , and Latin indices $i, j = 1, \dots, D - 2$ label the spatial directions within these planes.

2 Mass and tensions

In this section, we find expressions for the ADM mass and tensions in terms of the fall-off coefficients c_α and c_r that characterize the asymptotic form of the metric in (3). We begin by reviewing the ingredients of this calculations in asymptotically AdS spacetimes following [32, 17]. We then evaluate the mass and tensions for the planar AdS black hole metric and the AdS soliton [3], which will serve as important examples.

2.1 ADM charges

An ADM gravitational charge $Q(\xi)$ is defined with respect to an asymptotic symmetry ξ^a of a spacetime [32]. The first step in the computation is then to specify the boundary conditions on the metric and hence the asymptotic symmetries. As noted, we consider spacetimes that are asymptotically AdS in Poincare coordinates, with the rate of fall-off specified by equation (3). These conditions are analogous to standard asymptotically flat boundary conditions in the sense that they give finite mass and tensions, with values that are independent of the choice of slice at infinity used to compute the charge. Other choices of asymptotics for AdS may also be of interest, such as including logarithmic terms in the far field, or allowing the fall off coefficients to depend on the planar coordinates, but we defer consideration of these to future work.

The construction of ADM charges can be nicely expressed in the Hamiltonian formalism [33]. We will follow the particular treatment given in [34], which we briefly recount here. Let the spacetime be foliated by a family of hypersurfaces \mathcal{S} with unit normal n^a , and decompose the metric as

$$g_{ab} = (n \cdot n)n_a n_b + s_{ab} \ , \quad n \cdot n = \pm 1 \ , \quad n^a s_{ab} = 0, \quad (7)$$

where s_{ab} is the induced metric on \mathcal{S} with conjugate momentum π^{ab} . Let \bar{s}_{ab} and $\bar{\pi}^{ab}$ denote the metric and momentum in AdS, and let $h_{ab} = s_{ab} - \bar{s}_{ab}$ and $p^{ab} = \pi^{ab} - \bar{\pi}^{ab}$ be the differences between the metric and momenta and their asymptotic AdS values. Let ξ^a be a Killing field of AdS, and hence an asymptotic Killing vector of g_{ab} , which can be decomposed as

$$\xi^a = F n^a + \beta^a \ , \quad n_c \beta^c = 0. \quad (8)$$

The ADM gravitational charge associated with ξ^a is then defined by an integral over the

$(D-2)$ -dimensional boundary of \mathcal{S} in the asymptotic region,

$$Q(\xi) = -\frac{1}{16\pi} \int_{\partial\mathcal{S}_\infty} da_c B^c \quad (9)$$

where the boundary vector B^a is given by

$$B^a = F(D^a h - D_b h^{ab}) - h D^a F + h^{ab} D_b F + \frac{1}{\sqrt{s}} \beta^b (\bar{\pi}^{cd} h_{cd} \bar{s}^a{}_b - 2\bar{\pi}^{ac} h_{bc} - 2p^a{}_b) \quad (10)$$

and D_a is the derivative operator compatible with the background AdS metric \bar{s}_{ab} .

The ADM mass \mathcal{M} is the conserved charge associated with the time-translation symmetry of the asymptotic AdS spacetime. To obtain the explicit expression for the mass, one takes \mathcal{S} to be a constant time slice and the Killing field in (9) to be $\xi^a = (\partial/\partial t)^a$. In terms of the coordinates in (3), the asymptotic behavior of the normal form is given by $n_a = -(r/l)\nabla_a t$, while the area element is $da_c = d^{D-2}x (r/l)^{D-3} \nabla_c r$. From the definition (8) we have that $F = r/l$. Consequently, unlike in asymptotically flat spacetimes where F is a constant, the terms in (10) with derivatives acting on F contribute to the mass in AdS. The components of h_{ab} can be read off from equation (3), and recalling that with this choice of foliation $h_{tt} \equiv 0$, one finds that the ADM mass is given by

$$\mathcal{M} = \frac{\bar{v}}{16\pi l^D} \left((D-2)c_r + (D-1) \sum_{i=1}^{D-2} c_i \right), \quad (11)$$

where $\bar{v} = \prod_k L_k$ is the volume of the planar box at infinity.

The tension \mathcal{T}_i is the ADM charge associated with the spatial translation symmetry of the asymptotic AdS metric in the x^i -direction [34, 35], and hence the Killing vector in (9) is taken to be $\xi^a = (\partial/\partial x^i)^a$. The computation of \mathcal{T}_i is similar to that above for the mass, except that in this case the foliation is taken to be a family of constant spatial coordinate x^i . The boundary integral (9) then includes a trivial integration over time, which we divide out, so that \mathcal{T}_i is properly a tension per unit time. The boundary integral (10) then gives

$$L_j \mathcal{T}_j = \frac{\bar{v}}{16\pi l^D} \left((D-2)c_r + (D-1) \left[\sum_{i \neq j}^{D-2} c_i - c_t \right] \right) \quad (12)$$

with no sum on j implied on the left hand side.

2.2 Planar black hole and AdS soliton

In this section, we use the expressions obtained above to compute the ADM mass and tensions for the planar AdS black hole and AdS soliton [3]. The planar black hole metric is given by

$$ds^2 = -\frac{r^2}{l^2} \left(1 - \frac{r_0^{D-1}}{r^{D-1}} \right) dt^2 + \frac{r^2}{l^2} \sum_{i=1}^{D-2} (dx^i)^2 + \frac{l^2}{r^2} \left(1 - \frac{r_0^{D-1}}{r^{D-1}} \right)^{-1} dr^2 \quad (13)$$

with the horizon being located at $r = r_0$. Substitution into equations (11) and (12) yields for the planar black hole

$$\mathcal{M} = \frac{(D-2)\bar{v}r_0^{D-1}}{16\pi l^D}, \quad L_i \mathcal{T}_i = -\frac{\bar{v}r_0^{D-1}}{16\pi l^D} \quad (14)$$

where $i = 1, \dots, D-2$. The AdS soliton [3] is obtained from the black hole metric above via a double analytic continuation in time and one of the planar coordinates, yielding the metric

$$ds^2 = \frac{r^2}{l^2} \left(-dt^2 + \sum_{i=1}^{D-3} (dx^i)^2 + \left(1 - \frac{r_0^{D-1}}{r^{D-1}} \right) dz^2 \right) + \frac{l^2}{r^2} \left(1 - \frac{r_0^{D-1}}{r^{D-1}} \right)^{-1} dr^2 \quad (15)$$

Instead of a horizon, the spacelike Killing field ($\partial/\partial z$) has a fixed point at $r = r_0$. To avoid a conical singularity at the fixed point, the coordinate z must be identified with period $L_z = 4\pi l^2/(D-1)r_0$ [3]. The structure and thermodynamics of such bubbles, as well as spacetimes containing combinations of bubbles and black holes, have been studied extensively in the $\Lambda = 0$ case [14, 26, 16]. Using (11) and (12) one finds that the gravitational charges for the AdS soliton are given by

$$L_z \mathcal{T}_z = \frac{(D-2)\bar{v}r_0^{D-1}}{16\pi l^D}, \quad \mathcal{M} = L_i \mathcal{T}_i = -\frac{\bar{v}r_0^{D-1}}{16\pi l^D} \quad (16)$$

where now $i = 1, \dots, D-3$. One sees that the set of charges for the black hole and for the soliton are related via interchanging \mathcal{M} and $L_z \mathcal{T}_z$. In particular, the soliton has a negative mass but positive tension in the z -direction, while the black hole has positive mass and all tensions negative. In the next section we will show that the sum of the mass and the tensions is always zero for solutions, so some of these quantities, in fact, needed to have opposite signs.

3 Technical matters

3.1 Trace condition and Komar expressions

We now have the ADM expressions for the mass and tensions. However, the Smarr formulas result from applications of Komar integrals that naturally yield the equivalent Komar expressions for these quantities, the two sets of expressions being related via the field equations. This is familiar from the asymptotically flat case, where the ADM expression for the mass involves the fall-off coefficient c_r , while the Komar expression depends instead on c_t . The field equations in the asymptotic regime relate these two coefficients and yield the equivalence between the ADM and Komar expressions for the mass. We need to find the analogous relations, involving the fall-off coefficients c_i as well, with our planar AdS boundary conditions (3).

In order to do this, it is sufficient to expand the trace of the field equations to first order about the AdS metric in the asymptotic region. Because the cosmological term in (2) does

not contribute to the trace, we are looking simply at the linearization of $R = 0$ around AdS. Writing $g_{ab} = \bar{g}_{ab} + \gamma_{ab}$, with \bar{g}_{ab} the AdS metric, we then have

$$-\bar{\nabla}_a \bar{\nabla}^a \gamma + \bar{\nabla}_a \bar{\nabla}_b \gamma^{ab} - \bar{R}_{ab} \gamma^{ab} = 0 \quad (17)$$

where $\bar{\nabla}_a$ is the AdS covariant derivative operator, indices are raised and lowered with the AdS metric and $\bar{R}_{ab} = 2\Lambda \bar{g}_{ab}/(D-2)$. Upon substitution of the asymptotic form of the metric (3) into this equation, each of the first two terms on the left hand side vanishes identically, reducing the equation to $\Lambda \bar{g}^{ab} \gamma_{ab} = 0$. In terms of the fall-off coefficients this implies that

$$-c_t + c_r + \sum_{i=1}^{D-2} c_i = 0. \quad (18)$$

One readily checks that the planar black hole and AdS soliton solutions in Section (2.2) satisfy this condition⁷.

With the relation (18) in hand, we can eliminate the fall-off coefficient c_r from the expressions for the $(D-1)$ charges in (11) and (12) yielding the Komar-type expressions

$$M = \frac{\bar{v}}{16\pi l^D} \left((D-2)c_t + \sum_i c_i \right) \quad (19)$$

$$L_j \mathcal{T}_j = \frac{\bar{v}}{16\pi l^D} \left(-(D-2)c_j + \sum_{i \neq j} c_i - c_t \right) \quad (20)$$

with $i, j = 1, \dots, D-2$. From these expressions one can read off an important property of asymptotically planar AdS spacetimes, namely that the sum of the mass plus tensions of solutions necessarily vanishes for such spacetimes,

$$\mathcal{M} + \sum_i L_i \mathcal{T}_i = 0. \quad (21)$$

Equation (21), which is not readily apparent from formulas (11) and (12), can be viewed as a kind of Smarr relation that holds on solutions with planar AdS asymptotics, although non-standard in the sense that it relates only the ADM charges, without involving the interior geometry of the spacetime. Note that if we regard the mass and tensions as the diagonal elements of an ADM stress tensor in the planar directions, then (21) has the form of a zero trace condition.

This relation is imposed by the AdS asymptotics. It has no analogue, for example, in Kaluza-Klein theory with $\Lambda = 0$, where the mass and tensions are independent quantities [14, 15]. In contrast, we see that in AdS the mass and tensions are not independent. We expect that the zero trace condition (21) should in some way be a consequence of the conformal isometry of AdS generated by the Killing field $V^a = \sum_\alpha x^\alpha (\partial/\partial x^\alpha) - r(\partial/\partial r)$. However this is not apparent from the calculation given here and will be pursued in future work.

⁷A second, instructive way to derive equation (18) is in terms of the far field limit of the Killing potentials introduced in Section (4) below. This alternative demonstration of (18) is presented in Appendix A.

3.2 Gauge fixing

The task at hand is now to invert the formulas in (19) and (20) to give expressions for the fall-off coefficients c_t and c_i in terms of the ADM mass and tensions. If one regards these formulas as a single vector equation relating the vector of charges (M, \mathcal{T}_i) in terms of the vector of fall-off coefficients (c_t, c_i) , then one finds that the matrix has vanishing determinant, reflecting the fact that the charges must end up satisfying the constraint (21). In order to proceed, one charge must be removed from the vector and considered to be determined in terms of the others through equation (21). We will take \mathcal{T}_1 to be this auxiliary tension, and proceed with the reduced vector of $D - 2$ charges.

However, this leaves the counting off. There are now only $(D - 2)$ charges, but there are still $(D - 1)$ coefficients, and therefore the matrix is no longer square. The underlying problem is that the coordinates have not been completely fixed by the asymptotic form of the metric in equation (3). One can still perform a coordinate transformation that maintains the form of (3). This residual gauge freedom can be fixed *e.g.* by taking one of the fall-off coefficients to vanish, an approach that is similar to employing synchronous gauge in cosmology. We will take $c_{D-2} = 0$, which is convenient since the metrics given above for the planar black hole (13) and AdS soliton (15) already have this property. This gauge choice may be implemented via the coordinate transformation $x'^a = x^a + \chi^a$ where $\chi_r = kr^{-D}$ with k constant, and the remaining components $\chi_\alpha = 0$. It follows that $\nabla_r \chi_r = -(D - 1)kr^{-(D+1)}$ and that $\nabla_\alpha \chi_\beta + \nabla_\beta \chi_\alpha = (2k/l^4)\eta_{\alpha\beta}r^{-D+3}$. Hence, the constant k may be chosen such that the coefficient $c_{D-2} = 0$ in the new coordinates, while the other coefficients are merely relabeled. As a check, direct substitution shows that the expressions for the ADM charges are invariant under the coordinate transformation.

The reduced set of equations obtained by excluding the relation for $L_1 \mathcal{T}_1$ and setting $c_{D-2} = 0$ in (19) and (20) can now be inverted to give

$$\begin{aligned} c_t &= \frac{16\pi l^D}{(D-1)\bar{v}} (\mathcal{M} - L_{D-2} \mathcal{T}_{D-2}) \\ c_i &= \frac{16\pi l^D}{(D-1)\bar{v}} (L_{D-2} \mathcal{T}_{D-2} - L_i \mathcal{T}_i), \end{aligned} \tag{22}$$

where $i = 1, \dots, (D - 2)$.

3.3 Comparison with AdS/CFT boundary stress tensor

In this section we compare the expressions we have found for the ADM gravitational charges of asymptotically planar AdS spacetimes to those computed using the quasilocal boundary stress tensors method [36, 37, 38]. From the point of view of AdS/CFT one expects that, in the absence of a conformal anomaly, the boundary stress-energy tensor should be traceless. For definiteness, let us consider the five-dimensional case where the boundary stress tensor $\tau_{\alpha\beta}$ is given by [37]

$$8\pi\tau_{\alpha\beta} = \Theta_{\alpha\beta} - \left(\Theta + \frac{3}{l} + \frac{l}{4}R \right) h_{\alpha\beta} - \frac{l}{2}R_{\alpha\beta}, \tag{23}$$

where $h_{\alpha\beta}$ is the induced metric on a hypersurface of constant r in the asymptotic region, $\Theta_{\alpha\beta}$ is the extrinsic curvature of this slice, and $R_{\alpha\beta}$ is the Ricci tensor for the metric $h_{\alpha\beta}$. The leading contributions to $R_{\alpha\beta}$ arise, in principle, from second order terms in an expansion around infinity. However, these contributions vanish for the planar geometry [37]. Using the asymptotic form of the metric in (3) the extrinsic curvature components are found to be

$$\Theta_{tt} = \frac{r^2}{l^3} + \frac{1}{l^3 r^2} \left(c_t - \frac{c_r}{2} \right), \quad \Theta_{ij} = \left(-\frac{r^2}{l^3} + \frac{1}{l^3 r^2} \left(1 + \frac{c_r}{2} \right) \right) \delta_{ij} \quad (24)$$

These terms include divergent contributions from the AdS background that are cancelled by the $3h_{\alpha\beta}/l$ term in (23). The components of the boundary stress tensor are then given by

$$\tau_{tt} = \frac{1}{16\pi l^3 r^2} \left(3c_r + 4 \sum_i c_i \right), \quad \tau_{ij} = \frac{1}{16\pi l^3 r^2} \left(4c_t - 4 \sum_{k \neq i} c_k - 3c_r \right) \delta_{ij} \quad (25)$$

These match respectively with the integrands that gave rise to the expressions for the ADM mass and tensions given in (11) and (12) for $D = 5$. The boundary stress tensor charges are defined for Killing fields k^a of the boundary metric. Decomposing the $D - 1$ dimensional metric according to

$$h_{ab} = (n \cdot n) n_a n_b + \sigma_{ab} \quad n \cdot n = \pm 1 \quad n^a \sigma_{ab} = 0 \quad (26)$$

with σ_{ab} the metric on a $(D - 2)$ dimensional surface contained in the boundary. The charge is then defined as an integral over a $(D - 2)$ dimensional surface

$$Q(k)_{bdry} = \int_{\partial\Sigma} d^{D-2} x \sqrt{\sigma} (n^a \tau_{ab} k^b) \quad (27)$$

Since time and space translation Killing vectors of AdS are also Killing vectors of the boundary, this shows that the charges in the boundary formalism match the ADM quantities.

4 Deriving the Smarr formulas

In this section, we derive Smarr formulas for static asymptotically planar AdS spacetimes having $0 \leq p \leq (D - 2)$ spacelike Killing vectors that act as spatial translations at infinity, and which may contain black holes and/or bubbles. The first step is to write down a Komar integral identity associated with each Killing field, using the generalization of this procedure to $\Lambda \neq 0$ given in [28]. These Komar identities will relate the asymptotic behavior of the spacetime (3), to geometric properties of the horizon or bubble. The second step is to use equation (22) to convert the Komar identities into statements in terms of \mathcal{M} and \mathcal{T}_i . A related construction has appeared in reference [39], but focused only on the static Killing field.

Let g_{ab} be a metric satisfying the Einstein equation with cosmological constant (2) and let ξ^a be a Killing field of this metric. The generalization of the Komar integral identities to $\Lambda \neq 0$ in [28] involves an antisymmetric Killing potentials, satisfying

$$\nabla_a \omega^{ab} = \xi^b, \quad (28)$$

which necessarily exist because $\nabla_a \xi^a = 0$. Killing potentials are not unique. A solution to the homogeneous equation can always be added. However, this non-uniqueness does not affect the result, which comes about as follows. One starts from the differential identity $\nabla^a \nabla_a \xi^b = -R^b_a \xi^a$, which on solutions to the field equations can be rewritten as $\nabla^a \nabla_a \xi^b = -2\Lambda \nabla_c \omega^{cb}$. The Killing potential allows the right hand side to be cast as a total divergence, so that Stokes theorem can be used. Let the spacetime be foliated by surfaces \mathcal{S} with unit normal n_b as in (7), and let Σ to be a volume contained in \mathcal{S} with boundaries $\partial\Sigma_I$. Integrating the differential relation over Σ then gives the Komar integral identity [28]

$$\sum_I \int_{\partial\Sigma_I} ds_{ab} \left(\nabla^a \xi^b + \frac{2\Lambda}{D-2} \omega^{ab} \right) = 0 \quad (29)$$

where $ds_{ab} = \rho_{[a} n_{b]} da$ with ρ_a being the outward pointing unit normal to $\partial\Sigma_I$ within \mathcal{S} . There can be boundary contributions at infinity, black hole horizons, and bubbles, so that the Komar integral naturally splits up as

$$I_\infty + I_H + I_{bub} = 0 \quad (30)$$

where each of the terms corresponds to the integrals in (29) over a given type of boundary component. We proceed with this construction first for the static Killing vector and then for the spatial translation Killing fields. The derivation assumes that the black hole has a bifurcate horizon and that the spacelike Killing fields are tangent to the horizon.

4.1 Komar identity for the static Killing field

Let the Killing field ξ^a in the Komar integral identity (29) be the static Killing field, \mathcal{S} be a constant time slice which intersects the horizon at the bifurcation surface, and first consider the boundary integral at infinity. One finds that some individual terms in this are divergent, but that these divergences cancel when the full integrand in (29) is assembled⁸. This calculation proceeds as follows. Near infinity, one has $\rho_a n_b = -\nabla_a r \nabla_b t$ and the area element in (29) is therefore given by $da = (r/l)^{D-2} d^{D-2}x$. Finite contributions to the integrals will then come from terms that fall off as $r^{-(D-2)}$. One finds that

$$\rho_a n_b \nabla^a \xi^b = -\frac{r}{l^2} - \frac{1}{2l^2 r^{D-2}} ((D-2)c_t - 2c_r) \quad (31)$$

The first term on the right hand side will give a divergent contribution to the integral. This is cancelled [17] by a contribution from the Killing potential $\omega_{(t)}^{ab}$, where the label (t)

⁸Note that this has to be the case, if the integrals on the horizon and bubble are finite, since the Komar integral identity requires that these terms sum to zero.

indicates the corresponding Killing vector, [17]. The Killing potential may be written in the form $\omega_{(t)}^{ab} = \omega_{(t)AdS}^{ab} + \Delta\omega_{(t)}^{ab}$ where $\omega_{(t)AdS}^{ab}$ is the Killing potential in the background AdS spacetime, satisfying

$$\bar{\nabla}_a \omega_{(t)AdS}^{ab} = \xi^b \quad (32)$$

This can be taken to be $\omega_{(t)AdS}^{rt} = r/(D-1)$ and further recalling that $2\Lambda/(D-1)(D-2) = -1/l^2$ one has for the Killing potential term in (29)

$$\frac{2\Lambda}{D-2} \rho_a n_b \omega_{(t)}^{ab} = \frac{r}{l^2} + \frac{1}{2l^2 r^{D-2}} (-c_t + c_r) + \frac{2\Lambda}{D-2} \rho_a n_b \Delta\omega_{(t)}^{ab} \quad (33)$$

We see that the first term on the right hand side will cancel with the corresponding term in (31) giving an overall finite result for the boundary integral at infinity

$$I_\infty = \frac{\bar{v}}{2l^D} (2c_r - (D-1)c_t) + \frac{2\Lambda}{D-2} \int_{\partial\Sigma_\infty} ds_{ab} \Delta\omega_{(t)}^{ab} \quad (34)$$

where again the constant $\bar{v} = \prod_k L_k$.

We next turn to the horizon boundary term. Evaluation of the $\nabla_a \xi_b$ term at the horizon proceeds as in the $\Lambda = 0$ case [40], and is proportional to the surface gravity and combining this with the Killing potential term gives

$$I_H = \kappa_H A_H - \frac{2\Lambda}{D-2} \int_{\Sigma_H} ds_{ab} \omega_{(t)}^{ab} \quad (35)$$

The net contribution to (29) from the Killing potential will be the difference between its integrals at infinity and at the horizon, and it becomes useful to define quantities $\Theta_{(J)}$ for each Killing vector $\xi_{(J)}^a$, where $J = t, 1, \dots, p$, in terms of these differences by

$$\Theta_{(J)} = \sum_{in} \int_{\Sigma_{in}} ds_{ab} \omega_{(J)}^{ab} - \int_{\Sigma_\infty} ds_{ab} \Delta\omega_{(J)}^{ab} \quad (36)$$

where the sum is over interior boundaries, including both black hole horizons and bubbles. For the static Killing field there is no boundary contribution at the bubble, so long as it is smooth, and the Komar identity is then given by $I_\infty + I_H = 0$. Assembling the various ingredients then yields

$$\begin{aligned} \frac{\kappa_H A_H}{8\pi} - \frac{\Theta_{(t)} \Lambda}{4\pi(D-2)} &= \frac{\bar{v}}{16\pi l^D} ((D-1)c_t - 2c_r) \\ &= - \sum_{i=1}^{D-3} L_i \mathcal{T}_i \end{aligned} \quad (37)$$

where the second equality follows from using equations (18) and (22) to replace the fall-off coefficients with the gravitational charges. The fact that the sum over tensions only goes to $i = D-3$ rather than $i = D-2$ is due to the choice of gauge $c_{D-2} = 0$; examining equations (22) shows that \mathcal{T}_{D-2} drops out of the linear combination of the fall-off coefficients that appear in (37). The final Smarr formulas will not have this apparent gauge dependence, with one proviso to be noted below. When the spacetime is static but has no spatial Killing fields there is one puzzle, which we discuss at the end of Section (5).

4.2 Komar identities for spatial translations without fixed points

Now we turn to the Komar integral identities associated with the p spacelike Killing vectors, which we divide into two cases depending on whether the Killing field has a fixed point, or not. First consider the case when the Killing field $X^a = (\partial/\partial x)^a$ does not have a fixed point, and let the spacetime be foliated by slices \mathcal{S}_x of constant coordinate x , which are Lorentzian. Given the assumed absence of fixed points, there are again boundary terms only at infinity and at the black hole horizon. Each of these terms contains an integral over the time direction, which because the spacetime is assumed to be static, contributing only an overall factor of a time interval Δt . The evaluation of the boundary term at infinity is very similar to the calculation for the static Killing field. Near infinity $\rho_a x_b = \nabla_a r \nabla_b x$ and the area element is $da = (r/l)^{D-2} dt dx^{D-3}$ and as before the divergent term that arises from the Killing vector piece is canceled by the asymptotically AdS behavior of the Killing potential $\omega_{(x)}^{ab}$. Dividing through by the common factor of Δt and multiplying through by the length L_x , we find that the boundary term at infinity is given by

$$\left(\frac{L_x}{\Delta t}\right) I_\infty = -\frac{\bar{v}}{2l^D}((D-1)c_x + 2c_r) + \frac{2\Lambda}{D-2} \left(\frac{L_x}{\Delta t}\right) \int_{\partial\Sigma_\infty} ds_{ab} \Delta \omega_{(x)}^{ab} \quad (38)$$

In order to evaluate the boundary term at the horizon, we choose the slice such that the unit normal x_a is proportional to the Killing field, so that $X^a = Fx^a$. The boundary contribution from the derivative of the spatial Killing field in (29) can then be seen to vanish on the horizon, as follows. The generator ξ^a is normal to the horizon, so that the Killing vector term in the integrand of (29) is proportional to

$$x^b \xi^a \nabla_a X_b = -x^b \nabla_b (\xi^a X_a) + x^b X^a \nabla_b \xi_a. \quad (39)$$

The first term on the right hand side of this equation vanishes since $X_a \xi^a = 0$ on the horizon, and the derivative is tangent to the horizon. The second term vanishes because it is the contraction of an antisymmetric tensor and a symmetric tensor. For these vector fields there is no contribution from the bubble and therefore the Komar identity becomes

$$\begin{aligned} -\frac{\Lambda \Theta_{(x)}}{4\pi(D-2)} \left(\frac{L_x}{\Delta t}\right) &= \frac{\bar{v}}{16\pi l^D} ((D-1)c_x + 2c_r) \\ &= \mathcal{M} + \sum_{\substack{i=1 \\ i \neq x}}^{D-3} L_i \mathcal{T}_i \end{aligned} \quad (40)$$

where the quantity $\Theta_{(x)}$ is defined in equation (36). The Smarr formula (40) holds for any spacelike Killing field $(\partial/\partial x)^a$ with no fixed point.

4.3 Adding fixed points

The remaining case is when a spacelike Killing field generates a bubble, that is, $Z^a = (\partial/\partial z)^a$ has a fixed point on a $(D-2)$ -dimensional Lorentzian submanifold \mathcal{B} . A bolt in

the classification of Gibbons and Hawking [41], we refer to these as bubbles, following the work of [14, 26, 16]. The Killing field Z^a acts like a spatial translation around a compact dimension at infinity, but like a rotation in the vicinity of the bubble. The AdS soliton metric (15) has a bubble at $r = r_0$, which has the topology of a $(D - 3)$ -dimensional spatial torus cross time. When we refer to the area of a bubble, we will mean the area of the spatial torus at fixed time. The derivative of the Killing field Z_a at the bubble is normal to the bubble, so that

$$\nabla_{[a} Z_{b]} = \kappa_B (e_a^{(1)} e_b^{(2)} - e_a^{(2)} e_b^{(1)}) \quad (41)$$

where $e_a^{(1)}, e_a^{(2)}$ are a mutually orthonormal set of normal forms to \mathcal{B} [41] and κ_B is constant on the bubble [16]. It follows from (41) that $\kappa_B^2 = (1/2)(\nabla_a Z_b)\nabla^a Z^b$, which except for the sign is the identical to the formula for the surface gravity of a black hole horizon. The AdS soliton (15) has $\kappa_B = (D - 1)r_0/2l^2$. The geometry at the bubble is the same as that of a Euclidean black hole with the constant κ_B playing the role of surface gravity. Smoothness of the Euclidean black hole determines the periodicity in Euclidean time and hence its temperature. Smoothness of the bubble geometry requires that the periodicity of the z -coordinate be such that $\kappa_B L_z = 2\pi$.

We are now prepared to evaluate the Komar integral expression (29) for spacelike Killing vectors having fixed points. The boundary terms at the horizon and at infinity are unchanged from the case with no fixed point, with the only new contribution coming from I_B . Equation (41) implies that $e_a^{(1)} e_b^{(2)} \nabla^a Z^b = \kappa_B$, and we arrive at the expression

$$I_B = -A_B \kappa_B \Delta t + \frac{2\Lambda}{D-2} \int_{\partial\Sigma_\infty} ds_{ab} \Delta\omega_{(z)}^{ab}. \quad (42)$$

where A_B denotes the area of a $(D - 3)$ -dimensional spatial torus. Combining the three boundary terms, multiplying by L_z and dividing out a common factor of Δt gives the Smarr relation for a spacelike Killing field with fixed point

$$\begin{aligned} -\frac{\kappa_B L_z A_B}{8\pi} - \frac{\Lambda \Theta_{(z)}}{4\pi(D-2)} \left(\frac{L_z}{\Delta t} \right) &= \frac{\bar{v}}{16\pi l^D} ((D-1)c_z + 2c_r) \\ &= \mathcal{M} + \sum_{\substack{i=1 \\ i \neq z}}^{D-3} L_i \mathcal{T}_i \end{aligned} \quad (43)$$

where $\Theta_{(z)}$ is defined in (36). Although we are primarily interested in smooth bubbles, the derivation allows for a possible angular deficit $\psi = 2\pi - \kappa_B L_z$ at the bubble [16]. As long as $0 < \psi \leq 1$ this is a conical singularity with positive integrated curvature and corresponds to a $(D - 3)$ -brane wrapping the bubble.

5 Organizing the results

The purpose of this section is to rewrite the set of Smarr relations (37), (40) and (43) in a more transparent form. We start by finding conditions such that the different thermodynamic potentials $\Theta_{(J)}$ are all proportional to a common thermodynamic volume V .

5.1 Thermodynamic potentials and volumes

It was shown in [17] that the quantity $\Theta_{(t)}$ appearing in the Smarr relation (37) has a simple interpretation as minus the volume occupied by the black hole. A similar interpretation may be applied to all the potentials $\Theta_{(J)}$ appearing in (40) and (43). The boundary integral expression for $\Theta_{(J)}$ in (36) can be converted to a volume integral by means of Gauss's law. Choose the slice connecting the relevant boundaries such that its normal is in the direction of the Killing field, *i.e.* so that $\xi_{(J)}^a = F_{(J)} n_{(J)}^a$. Making use of the decompositions (7) and (8), it then follows that for each J one has $\int_{\partial\Sigma} \rho_a n_b \omega^{ab} = n \cdot n \int_{\Sigma} F \sqrt{|s|}$. Since the integrand at infinity in (36) depends on the difference between the Killing potentials for the metric g_{ab} and for the AdS metric \bar{g}_{ab} , the volume expression for $\Theta_{(J)}$ is likewise the difference between two integrals

$$\Theta_{(J)} = n_{(J)} \cdot n_{(J)} \left[\int_{\Sigma_{J,AdS}} F_{(J,AdS)} \sqrt{|s_{J,AdS}|} - \int_{\Sigma_{(J)}} F_{(J)} \sqrt{|s_{(J)}|} \right] \quad (44)$$

Here, the pure AdS volume integral extends inward all the way to $r = 0$, while the other volume integral extends only down to a black hole horizon or soliton bubble radius. For this reason, it was argued in [17, 22] that the quantity $-\Theta_{(t)}$ may be thought of as the volume occupied by the black hole. The properties of this thermodynamic volume were further explored in [18] where it was found that for rotating black holes the thermodynamic volume differs from the geometric volume in such a way that suggests a ‘reverse isoperimetric inequality’ for AdS black holes [18].

When the index J in (44) refers to one of the spatial Killing vectors, one has $n \cdot n = +1$ and the potential $\Theta_{(i)}$ is positive, with the quantity $L_i \Theta_{(i)} / \Delta t$ corresponding to a $(D - 1)$ -dimensional spatial volume. It is useful to define in each of these cases the positive volumes $V_{(J)}$ according to

$$V_{(t)} = -\Theta_{(t)}, \quad V_{(i)} = \left(\frac{L_i}{\Delta t} \right) \Theta_{(i)} \quad (45)$$

with $i = 1, \dots, p$. For the planar black hole and the AdS soliton spacetimes these volumes are redundant quantities. Due to symmetries and the diagonal form of the metric all the $V_{(J)}$ are equal to a common volume V ,

$$V_{(J)} = V \quad (46)$$

for $J = t, 1, \dots, p$, where in these cases we have $p = (D - 2)$. This raises the question as to how many independent volumes there are in general? We will show next that if two of the Killing vectors commute, then their associated volumes are necessarily equal. Therefore, a sufficient condition that all the thermodynamic volumes $V_{(J)}$ in the Smarr relations are equal to a common volume is that the set of Killing vectors be mutually commuting.

To demonstrate this, we need to show that if two of the Killing vectors commute, that the quantity $F_{(J)} \sqrt{|s_{(J)}|}$ will be the same for both vectors. So, assume that ξ^a and η^a are two commuting Killing vectors of an asymptotically planar AdS spacetime, with $\xi \simeq \partial/\partial t$

and $\eta \simeq \partial/\partial y$ in the asymptotic region. The area elements spanned by these vectors are then integrable and the metric can be put in block diagonal form (see *e.g.* the discussion in [42])

$$ds^2 = g_{tt}dt^2 + 2g_{ty}dtdy + g_{yy}dy^2 + \sum_{i,j=2}^{D-1} q_{ij}dx^i dx^j \quad (47)$$

where $q_{ij} = g_{ij}$. Now consider a constant time slice with normal $n_a = -F_{(t)}\nabla_a t$. If we decompose the Killing vector ξ^a according to (8), one has $\xi^a = F_{(t)}n^a + \beta^a$ where $g_{ty} = \beta_y$ and $g_{tt} = -(F_{(t)})^2 + \beta^y\beta_y$. This allows us to write the volume element for (47) as $\sqrt{-g} = F_{(t)}\sqrt{g_{yy}}\sqrt{q}$. One also has $\sqrt{s_{(t)}} = \sqrt{g_{yy}}\sqrt{q}$, which results in the statement $\sqrt{-g} = F_{(t)}\sqrt{s_{(t)}}$. These steps may be repeated for a slice of constant y , with the second Killing vector η^a decomposed to find that $\sqrt{-g} = F_{(y)}\sqrt{-g_{tt}}\sqrt{q} = F_{(y)}\sqrt{-s_{(y)}}$. Comparing the expressions for the volume elements from these two slicings establishes that $F_{(t)}\sqrt{|s_{(t)}|} = F_{(y)}\sqrt{|s_{(y)}|}$, which implies in turn that $\Theta_{(t)} = -(L_y/\Delta t)\Theta_{(y)}$ or equivalently the equality of the two thermodynamic volumes, $V_{(t)} = V_{(y)}$. Extending this argument to a larger set of commuting Killing vectors implies that the associated thermodynamic volumes will be equal.

5.2 Commuting Killing Fields

Equations (37), (40) and (43) give the Smarr relations that result from the static and spatial translation Killing fields. In this section we process these formulas to make their physical implications more apparent. We will assume that the spacetime has at least one freely acting spatial translation Killing field and that all the Killing fields commute, so that all the thermodynamic volumes are equal. At the end of the section we will briefly discuss the cases in which either the Killing fields do not commute, or there are no spatial symmetries.

For each freely acting spatial translation Killing field, the sum of equations (37) and (40) gives

$$M - L_{F_j}\mathcal{T}_{F_j} = \frac{\kappa_H A_H}{8\pi} \quad (48)$$

with $j = 1, \dots, n_F$ with n_F being the total number of freely acting spatial Killing vectors. Equation (48) implies that the product $L_{F_j}\mathcal{T}_{F_j}$ is the same for all the freely acting Killing vectors, *i.e.* that

$$L_{F_j}\mathcal{T}_{F_j} = (L\mathcal{T})_F \quad (49)$$

If there are also n_b spatial Killing vectors that having fixed points at a bubble, where necessarily $n_b + n_F = p$, then the sum of equations (37) and (43) gives for each B_i

$$M - L_{B_i}\mathcal{T}_{B_i} = \frac{\kappa_H A_H}{8\pi} - \frac{\kappa_{B_i} L_{B_i} A_{B_i}}{8\pi} \quad (50)$$

where $i = 1, \dots, n_b$ and A_{B_i} is the total area of bubbles⁹ corresponding to zeroes of the Killing vector $\partial/\partial x^i$. Recall that a smooth bubble has $\kappa_{B_i} L_{B_i} = 2\pi$.

⁹Multi-bubble solutions with $\Lambda = 0$ are presented in [14], although analogous solutions with $\Lambda \neq 0$ are not known at present.

In order to process the Smarr formula (37), we note that the left hand side is equal to $\mathcal{M} + L_{D-2}\mathcal{T}_{D-2}$. If we now assume that $\partial/\partial x^{D-2}$ is a freely acting Killing field¹⁰, recalling that by previous assumption there is at least one such Killing field, then it follows that $L_{D-2}\mathcal{T}_{D-2} = (L\mathcal{T})_F$ as in equation (48). Substitution for $L_{D-2}\mathcal{T}_{D-2}$ then yields the Smarr formula

$$\mathcal{M} = \frac{\kappa_H A_H}{8\pi} + \frac{\Lambda V}{8\pi(D-2)} \quad (51)$$

given in the introduction. These Smarr formulas give geometrical insight into the values of the mass and tensions for the planar black hole and AdS soliton solutions given in equations (14) and (16). We see that \mathcal{T}_F is always negative (see (4)) and that \mathcal{M} is always greater than or equal to \mathcal{T}_F , because they differ by the horizon area. Substituting (48), (50), (51) into (37) and using the trace constraint additionally gives a formula for the sum of the $D-2-p$ tensions not associated with symmetry directions

$$- \sum_{not\ KV} L_i \mathcal{T}_i = \sum_{i=1}^{n_b} \frac{L_{B_i} \kappa_{B_i} A_{B_i}}{8\pi} + \frac{\kappa_H A_H}{8\pi} + \frac{(p+1)\Lambda V}{8\pi(D-2)}. \quad (52)$$

To summarize, the relations (48), (50), (51), and (52) hold for spacetimes that are static, with p spatial-translation Killing fields, at least one of which is freely acting, and assuming that all the Killing fields are mutually commuting. These Smarr formulas along with the zero trace constraint (21) are the main results of this paper.

The symmetry between bubbles and black holes noted in the introduction is apparent here. The equations are invariant if \mathcal{M} and $\kappa_H A_H$ are interchanged with \mathcal{T}_{B_i} and $\kappa_{B_i} L_{B_i} A_{B_i}$. In the AdS/CFT correspondence, the black hole corresponds to a high temperature, deconfined phase of the gauge theory, while the soliton corresponds to the low temperature confining phase [4, 5, 3, 7]. It would be interesting to understand the nature of the symmetry between these phases that is suggested by our results. One intriguing possibility allowed by the equations is a critical, or transition, spacetime containing a mix of bubbles and black holes such that $M = \mathcal{T}_B = 0$. In this case, the trace constraint would have to be satisfied by balancing the negative \mathcal{T}_F with positive non-symmetry \mathcal{T}_i .

5.3 More general case

When the Killing fields do not commute we need to allow for the possibility that the thermodynamic volumes are not equal. Repeating the steps of the previous subsection gives

$$M - L_{F_j} \mathcal{T}_{F_j} = \frac{\kappa_H A_H}{8\pi} + \frac{\Lambda(V_{(t)} - V_{(F_j)})}{4\pi(D-2)}, \quad (53)$$

¹⁰It appears to be inconsistent for $\partial/\partial x^{D-2}$ to be non-freely acting. In Section (3.2) we showed that it is always possible via a coordinate transformation in the asymptotic region to set one of the falloff coefficients, taken to be c_{D-2} , equal to zero. Such a coordinate transformation necessarily changes the components of the metric in the interior as well. If $\partial/\partial x^{D-2}$ were not freely acting, then both the metric and its normal derivatives would be fixed at the corresponding bubble. Hence trying to specify the behavior both at the bubble and at infinity will in general not be possible, as can be checked for the AdS soliton metric (16).

where $j = 1, \dots, n_F$ and additionally with $i = 1, \dots, n_b$

$$L_{B_i} \mathcal{T}_{B_i} - L_{F_j} \mathcal{T}_{F_j} = \frac{\kappa_{B_i} L_{B_i} A_{B_i}}{8\pi} + \frac{\Lambda (V_{(B_i)} - V_{(F_j)})}{4\pi(D-2)} \quad (54)$$

Equation (54) can be rearranged to put all the bubble terms with index B_i on one side, and the freely acting terms with index F_j on the other. Since this is true for all choices of i and j and noting an analogous property of equation (53) we learn that the combination

$$L_j \mathcal{T}_j - \frac{\Lambda V_{(j)}}{4\pi(D-2)} - \frac{L_j \kappa_j A_j}{8\pi} = \mathcal{M} - \frac{\kappa_H A_H}{8\pi} - \frac{\Lambda V_{(t)}}{4\pi(D-2)} \equiv \mathcal{C} \quad (55)$$

with the same constant value \mathcal{C} for all symmetry directions, and where the expression for $L_{F_j} \mathcal{T}_{F_j}$ is obtained by setting $A_j = 0$. We recall that in previous work on AdS black holes with spherical topology [17, 19, 20, 21] the combination $M - \Lambda V/4\pi(D-3)$ arose as a relevant thermodynamic quantity. For the planar geometry the dimension dependent coefficient changes and the corresponding combination is $M - \Lambda V_{(t)}/4\pi(D-2)$. Interestingly, we see that analogous quantities from the spatial symmetries also play a role in the brane thermodynamics, namely the combination $L_i \mathcal{T}_i - \Lambda V_{(i)}/4\pi(D-2)$. The space-time quantities $Q(\partial/\partial x^J) - \Lambda V_{(J)}/4\pi(D-2)$ altogether form a set of thermodynamic variables, where the $Q(\partial/\partial x^J)$ are the ADM charges associated with space and time translation symmetries. A question for future work is to study the properties of these quantities further, for example by computing the corresponding first laws governing their variations.

If the spacetime only has a static Killing field then equation (37) is the only Smarr relation. Using the trace constraint, this can be rewritten as

$$\mathcal{M} + L_{D-2} \mathcal{T}_{D-2} = \frac{\kappa_H A_H}{8\pi} + \frac{\Lambda V_{(t)}}{4\pi(D-2)} \quad (56)$$

This formula would apply, *e.g.* to a localized or “caged” black hole with spherical horizon topology. For example, suppose that all the periodicities L_i of the spatial directions are equal so that the black hole horizon is asymptotically spherically symmetric within each constant r plane. All the coefficients c_i would then be the same and all the products $L_i \mathcal{T}_i$ would be equal. The zero trace condition then implies that this common value is given by $-M/(D-2)$, so that equation (56) becomes

$$(D-3)\mathcal{M} = (D-2) \frac{\kappa_H A_H}{8\pi} + \frac{\Lambda V_{(t)}}{4\pi}, \quad (57)$$

which is the same as the Smarr relation as for a static AdS black hole with spherical horizon topology [17].

The Smarr relation (56) has the puzzling feature that $L_{D-2} \mathcal{T}_{D-2}$ is singled out, which can be traced back to the gauge choice $c_{D-2} = 0$. It seems to us that there is another piece of physics that needs to be used to get rid of this apparent gauge dependence. For example, we have just argued that the additional assumption of spherical symmetry asymptotically on constant r slices allows one to reach a more satisfying result. It is plausible that specifying the symmetries of the black hole allows one to more generally determine $L_{D-2} \mathcal{T}_{D-2}$ in terms of \mathcal{M} . This remains a question for future work.

6 Discussion and Conclusion

In this paper we have studied the implications of spatial translation symmetries for static AdS black hole and soliton spacetimes. The associated ADM charges, which are defined in the context of the Hamiltonian formulation of general relativity, agree with the corresponding components of the AdS boundary stress tensor. We have shown that the sum of the ADM mass and tensions of a solution is zero. The set of Smarr formulas obtained provide relations between the mass, tensions, horizon and bubble areas, as well as a set of thermodynamic volumes. These relations display a duality between black holes and bubbles, in which the mass and black hole area are interchanged with the bubble tension and bubble area. Given the interpretation as deconfined/confined phases of the dual gauge theory, it would be of interest to further understand how the black hole/bubble symmetry manifests in the boundary gauge theory. A related question is to sort out what the thermodynamic volumes correspond to in the CFT. Since in general the bulk Killing fields do not commute, it is tempting to conjecture that these “renormalized” volumes relate to integrals of (possibly non-commuting) gauge fields around loops in the boundary.

Another area for further research is to study more general asymptotics, for example, when the fall-off coefficients depend on the planar coordinates x^α . Further, the AdS soliton-black string solution given in [12] and the numerical droplet type black hole solutions given in [43] do not have the asymptotically AdS form of solutions considered here, since the black hole horizon extends to infinity. A question for future consideration is how to define the ADM charges of these spacetimes and to derive the corresponding Smarr relations.

7 Acknowledgments

This work was supported in part by NSF grant PHY-0555304. JT thanks the Centro de Ciencias de Benasque Pedro Pascual, where this work was initiated, for their hospitality.

A Appendix: Killing potential

We use perturbation theory to solve for the Killing potential in the asymptotic region of the spacetime. Recalling the definition of the antisymmetric Killing potential, one can write

$$\frac{1}{\sqrt{g}}\partial_a(\sqrt{g}\omega^{ab}) = K^b \quad . \quad (\text{A.58})$$

For definiteness, we take the Killing field to be the generator of time translations. The metric functions in (3) depends solely on r , and hence the Killing potential components are consequently assumed to depend only on r . We moreover write the asymptotic metric as a perturbative expansion off AdS, and assume a similar expansion to exist for ω^{rt}

$$g_{ab} = \bar{g}_{ab} + \kappa\gamma_{ab} + \mathcal{O}(\kappa^2) \quad , \quad \omega^{rt}(r) = \omega_{(0)}^{rt} + \kappa\omega_{(1)}^{rt} + \mathcal{O}(\kappa^2) \quad . \quad (\text{A.59})$$

Expanding the metric determinant then yields

$$\sqrt{-g} = \sqrt{-\bar{g}} \left(1 + \kappa \frac{\gamma}{2} + \mathcal{O}(\kappa^2) \right) \quad , \quad \gamma \equiv \bar{g}^{ab}\gamma_{ab} \quad . \quad (\text{A.60})$$

The solution for the zeroth order equation is that for pure AdS, so $\omega_{(1)}^{rt}$ is the solution to

$$\left(\frac{d}{dr} + \frac{D-2}{r} \right) \omega_{(1)}^{rt} = -\frac{r}{2(D-1)} \frac{d\gamma}{dr} \quad . \quad (\text{A.61})$$

From (3), we find $\gamma = \frac{c_r + \sum_i c_i - c_t}{2r^{D-1}}$, and finally

$$\omega_{(1)}^{rt} = \frac{c_r + \sum_i c_i - c_t}{2r^{D-2}} \ln r + \frac{a}{r^{D-2}}, \quad (\text{A.62})$$

Where a is an integration constant to be determined by appropriate boundary conditions on the Killing potential. Recalling that the leading contribution to the area element in the Komar integral goes as $r^{-(D-2)}$, we see that the first term in the above equation diverges logarithmically at infinity. Since the Komar integral relation is a rewriting of Einstein equation, the sum of the fall-off coefficients has to vanish to have solutions with the assumed asymptotics. That is, the trace of the metric perturbations vanishes on solutions,

$$c_r + \sum_{i=1}^{D-2} c_i - c_t = 0 \quad (\text{A.63})$$

References

- [1] J. P. S. Lemos, ‘‘Cylindrical black hole in general relativity,’’ Phys. Lett. B **353**, 46 (1995) [gr-qc/9404041].
- [2] R. -G. Cai and Y. -Z. Zhang, ‘‘Black plane solutions in four-dimensional space-times,’’ Phys. Rev. D **54**, 4891 (1996) [gr-qc/9609065].

- [3] G. T. Horowitz and R. C. Myers, “The AdS / CFT correspondence and a new positive energy conjecture for general relativity,” *Phys. Rev. D* **59**, 026005 (1998) [hep-th/9808079].
- [4] E. Witten, “Anti-de Sitter space, thermal phase transition, and confinement in gauge theories,” *Adv. Theor. Math. Phys.* **2**, 505 (1998) [hep-th/9803131].
- [5] E. Witten, “Anti-de Sitter space and holography,” *Adv. Theor. Math. Phys.* **2**, 253 (1998) [hep-th/9802150].
- [6] C. S. Peca and J. P. S. Lemos, “Thermodynamics of toroidal black holes,” *J. Math. Phys.* **41**, 4783 (2000) [gr-qc/9809029].
- [7] S. Surya, K. Schleich and D. M. Witt, “Phase transitions for flat AdS black holes,” *Phys. Rev. Lett.* **86**, 5231 (2001) [hep-th/0101134].
- [8] S. W. Hawking and D. N. Page, “Thermodynamics of Black Holes in anti-De Sitter Space,” *Commun. Math. Phys.* **87**, 577 (1983).
- [9] R. de Mello Koch, A. Jevicki, M. Mihailescu and J. P. Nunes, “Evaluation of glueball masses from supergravity,” *Phys. Rev. D* **58**, 105009 (1998) [hep-th/9806125].
- [10] G. T. Horowitz and V. E. Hubeny, “Quasinormal modes of AdS black holes and the approach to thermal equilibrium,” *Phys. Rev. D* **62**, 024027 (2000) [hep-th/9909056].
- [11] V. E. Hubeny, D. Marolf and M. Rangamani, “Hawking radiation from AdS black holes,” *Class. Quant. Grav.* **27**, 095018 (2010) [arXiv:0911.4144 [hep-th]].
- [12] F. M. Haehl, “The Schwarzschild-Black String AdS Soliton: Instability and Holographic Heat Transport,” [arXiv:1210.5763 [hep-th]].
- [13] T. Harmark and N. A. Obers, “New phase diagram for black holes and strings on cylinders,” *Class. Quant. Grav.* **21**, 1709 (2004) [hep-th/0309116].
- [14] H. Elvang, T. Harmark and N. A. Obers, “Sequences of bubbles and holes: New phases of Kaluza-Klein black holes,” *JHEP* **0501**, 003 (2005) [hep-th/0407050].
- [15] D. Kastor and J. Traschen, “Stresses and Strains in the First Law for Kaluza-Klein Black Holes,” *JHEP* **0609**, 022 (2006) [hep-th/0607051].
- [16] D. Kastor, S. Ray and J. Traschen, “The Thermodynamics of Kaluza-Klein Black Hole/Bubble Chains,” *Class. Quant. Grav.* **25**, 125004 (2008) [arXiv:0803.2019 [hep-th]].
- [17] D. Kastor, S. Ray and J. Traschen, “Enthalpy and the Mechanics of AdS Black Holes,” *Class. Quant. Grav.* **26**, 195011 (2009) [arXiv:0904.2765 [hep-th]].

- [18] M. Cvetič, G. W. Gibbons, D. Kubiznak and C. N. Pope, “Black Hole Enthalpy and an Entropy Inequality for the Thermodynamic Volume,” *Phys. Rev. D* **84**, 024037 (2011) [arXiv:1012.2888 [hep-th]].
- [19] D. Kubiznak and R. B. Mann, “P-V criticality of charged AdS black holes,” *JHEP* **1207**, 033 (2012) [arXiv:1205.0559 [hep-th]].
- [20] B. P. Dolan, “Where is the PdV term in the first law of black hole thermodynamics?,” arXiv:1209.1272 [gr-qc].
- [21] B. P. Dolan, “Pressure and volume in the first law of black hole thermodynamics,” *Class. Quant. Grav.* **28**, 235017 (2011) [arXiv:1106.6260 [gr-qc]].
- [22] B. P. Dolan, “Enthalpy and the first law of black hole thermodynamics,” *PoS CNCFG* **2010**, 017 (2010).
- [23] B. P. Dolan, D. Kastor, D. Kubiznak, R. B. Mann and J. Traschen, “Thermodynamic Volumes and Isoperimetric Inequalities for de Sitter Black Holes,” arXiv:1301.5926 [hep-th].
- [24] R. L. Arnowitt, S. Deser and C. W. Misner, “The Dynamics of general relativity,” gr-qc/0405109.
- [25] L. Smarr, “Mass formula for Kerr black holes,” *Phys. Rev. Lett.* **30**, 71 (1973) [Erratum-ibid. **30**, 521 (1973)].
- [26] H. Elvang and G. T. Horowitz, “When black holes meet Kaluza-Klein bubbles,” *Phys. Rev. D* **67**, 044015 (2003) [hep-th/0210303].
- [27] S. L. Bazanski and P. Zyla, “A Gauss type law for gravity with a cosmological constant,” *Gen. Rel. Grav.* **22**, 379 (1990).
- [28] D. Kastor, “Komar Integrals in Higher (and Lower) Derivative Gravity,” *Class. Quant. Grav.* **25**, 175007 (2008) [arXiv:0804.1832 [hep-th]].
- [29] R. Schon and S. -T. Yau, “Proof of the positive mass theorem. 2.,” *Commun. Math. Phys.* **79**, 231 (1981).
- [30] E. Witten, “A Simple Proof of the Positive Energy Theorem,” *Commun. Math. Phys.* **80**, 381 (1981).
- [31] J. H. Traschen, “A Positivity theorem for gravitational tension in brane space-times,” *Class. Quant. Grav.* **21**, 1343 (2004) [hep-th/0308173].
- [32] L. F. Abbott and S. Deser, “Stability of Gravity with a Cosmological Constant,” *Nucl. Phys. B* **195**, 76 (1982).

- [33] T. Regge and C. Teitelboim, “Role of Surface Integrals in the Hamiltonian Formulation of General Relativity,” *Annals Phys.* **88**, 286 (1974).
- [34] J. H. Traschen and D. Fox, “Tension perturbations of black brane space-times,” *Class. Quant. Grav.* **21**, 289 (2004) [gr-qc/0103106].
- [35] P. K. Townsend and M. Zamaklar, “The First law of black brane mechanics,” *Class. Quant. Grav.* **18**, 5269 (2001) [hep-th/0107228].
- [36] J. D. Brown and M. Henneaux, “Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity,” *Commun. Math. Phys.* **104**, 207 (1986).
- [37] V. Balasubramanian and P. Kraus, “A Stress tensor for Anti-de Sitter gravity,” *Commun. Math. Phys.* **208**, 413 (1999) [hep-th/9902121].
- [38] R. C. Myers, “Stress tensors and Casimir energies in the AdS / CFT correspondence,” *Phys. Rev. D* **60**, 046002 (1999) [hep-th/9903203].
- [39] I. Papadimitriou and K. Skenderis, “Thermodynamics of asymptotically locally AdS spacetimes,” *JHEP* **0508**, 004 (2005) [hep-th/0505190].
- [40] J. M. Bardeen, B. Carter and S. W. Hawking, “The Four laws of black hole mechanics,” *Commun. Math. Phys.* **31**, 161 (1973).
- [41] G.W. Gibbons and S.W. Hawking, “Classification of Gravitational Instanton Symmetries,” *Commun. Math. Phys.* **66**, 291 (1979).
- [42] B. Carter, “Mathematical Foundations of the Theory of Relativistic Stellar and Black Hole Configurations,” in “Gravitation in Astrophysics, Cargese 1986”, B. Carter and J. Hartle, eds, Plenum Press, N.Y., 1987.
- [43] P. Figueras, J. Lucietti and T. Wiseman, “Ricci solitons, Ricci flow, and strongly coupled CFT in the Schwarzschild Unruh or Boulware vacua,” *Class. Quant. Grav.* **28**, 215018 (2011) [arXiv:1104.4489 [hep-th]].